INDUCTIVE RING TOPOLOGIES(1)

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Since topological algebra is the study of algebraic structures with topologies for which the operations are continuous, a natural question for the topological algebraist to ask is whether a given structure admits any such topologies whatever, other than the discrete and indiscrete ones. The question has been answered for some classes of structures. For example, Kertész and Szele [7] prove that every infinite abelian group admits a nondiscrete, Hausdorff group topology. On the other hand, Hanson [5] gives an example of an infinite groupoid which admits only the two trivial topologies mentioned above.

Our purpose here is to answer this question for infinite fields, proving that every infinite field admits a nondiscrete, Hausdorff field topology. This will be done by affirmatively answering the question for two classes of commutative rings: the first being all integral domains with a certain cardinality condition (§3), and the second, all rings which are the union of a chain of subrings with certain properties (§4). These two classes will be shown to include all infinite fields (§5).

Our method of proof will make use of an inductive procedure first used by Hinrichs [6] to prove the existence of certain unusual topologies on the integers. The procedure is described in §1, where we define what we mean by an "inductive ring topology".

In §§7 and 8, we turn our attention to some further applications of inductive topologies, showing first how they can be used to construct interesting examples of topologies on the integers and rational numbers. We use them to get proofs that there are uncountably many, and non-first countable ring topologies on all the rings considered in §3 and §4. We also show how characterizations can be obtained for several classes of topologies on fields using modifications of the inductive method.

A supplement to our discussion of field topologies comes in §6, where we characterize those fields which admit nondiscrete, Hausdorff, locally bounded topologies. The methods used here, however, are those of valuation theory.

When we say that a topology $\mathscr{T}$ is a *ring topology* on a ring A, we mean that the mappings $(a, b) \to a-b$ and $(a, b) \to a \cdot b$ from $A \times A$ into A are continuous. $\mathscr{T}$ is a *field topology* on a field K if it is a ring topology, and in addition, the mapping $a \to a^{-1}$ is continuous on $K \sim \{0\}$.

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1. **Definitions of inductive topologies.** Let A be a commutative ring with identity. Then a unique first countable ring topology on A is determined if we take as a basic system of neighborhoods of zero a collection $\{V_n : n \geqq 0\}$ of subsets of A having the following properties for all $n \geqq 0$, [2, p. 76].

(1.1) $$0 \in V_n,$$

(1.2) $$V_n = -V_n,$$

(1.3) $$V_{n+1} + V_{n+1} \subseteq V_n,$$

(1.4) $$V_{n+1} \cdot V_{n+1} \subseteq V_n,$$

(1.5) For any x in A, there is an integer k such that $x \cdot V_{n+k} \subseteq V_n$.

Furthermore, the topology is Hausdorff if and only if [2, p. 14]

(1.6) $$\bigcap_{n=0}^{\infty} V_n = \{0\}.$$

Suppose now that $\{V_n : n \geqq 0\}$ satisfies (1.1) to (1.5), and that a_k is in V_k for each $k \geqq 1$. Then clearly for each m and n with $m > n$, by repeated applications of (1.3) and (1.4), one can show that V_n contains certain algebraic combinations of a_m, $a_{m-1}, \ldots, a_{n+1}$. From (1.5), we can see that certain multiples xa_k are in V_n for $n+1 \leqq k \leqq m$.

It is from these elementary observations that the idea for an inductive topology is derived. To get an inductive topology, we begin with a sequence $a_1, a_2, \ldots$, and inductively build up the sets $V_0, V_1, V_2, \ldots$ so that they contain only the algebraic combinations of $a_1, a_2, a_3, \ldots$ necessary so that (1.1)–(1.5) are satisfied. Let us now describe the procedure in detail.

Since the sets $V_0, V_1, V_2, \ldots$ will contain only polynomial expressions in $a_1, a_2, a_3, \ldots$, it will prove to be advantageous to at first replace this sequence of elements of A by a sequence of indeterminates $X_1, X_2, X_3, \ldots$. Let $A[(X_n)]$ denote the ring of polynomials over A in these indeterminates. Let $(B_k)_{k \geqq 1}$ be a sequence of subsets of A which satisfies the following conditions.

(1.7) $$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$$

(1.8) $D \cup -D$ multiplicatively generates A, $D = \bigcup_{n=1}^{\infty} B_n$.

A set S *multiplicatively generates* the ring A if every element of A is a product of elements of S.

We begin by defining a double sequence of sets of polynomials in $A[(X_n)]$:

$$\begin{array}{llll} W_0^0, & W_0^1, & W_0^2, & W_0^3, \ldots \\ & W_1^1, & W_1^2, & W_1^3, \ldots \\ & & W_2^2, & W_2^3, \ldots \\ & & & W_3^3, \ldots . \end{array}$$

Let W_0^0 be the set containing only the zero polynomial. That is,

$$W_0^0 = \{0\}. \tag{1.9}$$

Assume now that the sets W_n^m have been defined for each n and m such that $0 \leqq n \leqq m \leqq k$. Let

$$W_{k+1}^{k+1} = \{0, X_{k+1}, -X_{k+1}\}. \tag{1.10}$$

If W_j^{k+1} has been defined for each j such that $k+1 \geqq j \geqq r+1$, then define W_r^{k+1} by

$$\begin{aligned} W_r^{k+1} = &\left[\left(W_{r+1}^{k+1} + \bigcup_{s=r+1}^{k+1} W_{r+1}^s\right) \cup \left(W_{r+1}^{k+1} \cdot \bigcup_{s=r+1}^{k+1} W_{r+1}^s\right) \cup (B_{r+1} \cdot W_{r+1}^{k+1})\right] \\ &\sim \left[\bigcup_{s=r}^{k} W_r^s\right], \end{aligned} \tag{1.11}$$

where " $\sim$ " denotes relative complementation.

For each $n \geqq 0$, we define W_n to be the union of the sets in the nth row of the array. That is,

$$W_n = \bigcup_{m=n}^{\infty} W_n^m. \tag{1.12}$$

One may easily verify that we have built into the collection of sets $\{W_n : n \geqq 0\}$ the following properties for each $n \geqq 0$.

$$0 \in W_n, \tag{1.13}$$

$$W_n = -W_n, \tag{1.14}$$

$$W_{n+1} + W_{n+1} \subseteq W_n, \tag{1.15}$$

$$W_{n+1} \cdot W_{n+1} \subseteq W_n, \tag{1.16}$$

$$B_{n+1} \cdot W_{n+1} \subseteq W_n. \tag{1.17}$$

From properties (1.13)–(1.15), we see that the collection $\{W_n : n \geqq 0\}$ is a basic system of neighborhoods of zero for an additive group topology on $A[(X_n)]$. Indeed, one can see (Lemma 2.2) that the topology is Hausdorff. From property (1.16), we observe that multiplication is continuous at zero. From (1.17), we can derive the following generalization.

$$\begin{gathered} \text{For any } x \text{ in } A \text{ there is an integer } k \\ \text{such that for all } n \geqq 0, x \cdot W_{n+k} \subseteq W_n. \end{gathered} \tag{1.17$'$}$$

To see this, let x be any element of A. Then by (1.8), there are elements $x_1, x_2, \ldots, x_m$ in D such that $x = \pm x_1 x_2 \cdots x_m$. By (1.7), there is an integer k_0 such that $x_j \in B_{k_0}$ for all j such that $1 \leqq j \leqq m$. Let $k = k_0 + m$. Then clearly by (1.7), $x_{(m+1)-j} \in B_{n+(k-j)}$, where $1 \leqq j \leqq m$, and $n \geqq 0$, since $n+(k-j) \geqq k_0$. Thus for any $n \geqq 0$, by (1.17) and (1.14),

$$\begin{aligned} xW_{n+k} &= x_1x_2\cdots x_m W_{n+k} \subseteq x_1\cdots x_{m-1}W_{n+k-1} \\ &\subseteq \cdots \subseteq x_1\cdots x_{m-j}W_{n+k-j} \subseteq \cdots \subseteq x_1 W_{n+k-(m-1)} \\ &\subseteq W_{n+k-m} = W_{n+k_0} \subseteq W_n. \end{aligned}$$

We now derive a topology on A from this one on $A[(X_n)]$. Let $(a_k)_{k\geqq 1}$ be a sequence of elements of A. Let $\sigma_{(a_k)}$ be the substitution homomorphism from $A[(X_n)]$ into A defined by

$$\sigma_{(a_k)}: P(X_1, X_2, \ldots) \to P(a_1, a_2, \ldots)$$

for all polynomials P in $A[(X_n)]$. Then $\sigma_{(a_k)}$ is indeed a homomorphism from $A[(X_n)]$ into A, where the domain and range are regarded as algebras over A.

To get the desired neighborhoods of zero in A, for $0\leqq n\leqq m$, let

$$V_n^m = \sigma_{(a_k)}(W_n^m) \sim \bigcup_{j=n}^{m-1} \sigma_{(a_k)}(W_n^j), \tag{1.18}$$

$$V_n \doteq \sigma_{(a_k)}(W_n). \tag{1.19}$$

It is clear, then, that $V_n = \bigcup_{m=n}^{\infty} V_n^m$. Also, from (1.13)–(1.17′), and the fact that $\sigma_{(a_k)}$ is an algebra homomorphism, it follows that for all $n\geqq 0$, properties (1.1) to (1.5) hold. In addition, for all $n\geqq 0$,

$$B_{n+1}\cdot V_{n+1} \subseteq V_n. \tag{1.5′}$$

Thus, $\mathscr{V} = \{V_n : n\geqq 0\}$, is a basic system of neighborhoods of zero for a ring topology on A in which the sequence (a_k) converges to zero. Note that this is just the quotient topology on A determined by the mapping $\sigma_{(a_k)}$ and the topology given to $A[(X_n)]$.

Definition. Call the topology just defined the *inductive ring topology* on A determined by the sequences (a_k) and (B_k). Denote it by $\mathscr{T}((a_k), (B_k))$. Call the topology on $A[(X_n)]$ the *inductive polynomial topology* determined by the sequence (B_k), and denote it by $\mathscr{T}((B_k))$. For brevity, we will sometimes call an inductive ring topology simply an inductive topology.

We note here that even though an inductive polynomial topology $\mathscr{T}((B_k))$ is always Hausdorff, an inductive ring topology $\mathscr{T}((a_k), (B_k))$ derived from it need not be. In fact, $\mathscr{T}((a_k), (B_k))$ may be the indiscrete topology. We will be interested, then, in finding ways to suitably restrict A and choose the sequences (a_k) and (B_k) so that $\mathscr{T}((a_k), (B_k))$ can be proven to be Hausdorff.

2. Some basic lemmas.

The first two lemmas in this section identify the properties of the polynomials in the sets W_n^m which make possible the construction of Hausdorff topologies under suitable conditions. The third gives a useful sufficient condition for Hausdorff separation for inductive topologies.

For a polynomial P in $A[(X_n)]$, by the *monomials of* P, we will mean those monomials in P which have nonzero coefficients. A *monomial* of course is a product $cX_{k_1}^{r_1}X_{k_2}^{r_2}\cdots X_{k_n}^{r_n}$ of powers of finitely many indeterminates, with a coefficient c in A. For a polynomial P in $A[(X_n)]$, let $\deg_m(P)$ denote the degree of P in the indeterminate X_m.

For each P in $A[(X_n)]$, let P_m^* denote the polynomial which is the sum of the monomials of P not divisible by X_m. Let $^*P_m$ be the sum of the monomials of P which are divisible by X_m. It is clear then, that $P={}^*P_m+P_m^*$ for all $m\geqq 1$.

LEMMA 2.1. *Let P be in W_n^m. If $n<m$, then P_m^* is in W_n^j for some j such that $n\leqq j<m$.*

Proof. We use a double induction argument. The proposition holds vacuously for the set W_0^0. Suppose then it holds for all sets W_n^m where $n\leqq m\leqq k$. It holds vacuously for W_{k+1}^{k+1}, and is obvious for

$$W_k^{k+1} = [\{\pm X_{k+1},\ \pm 2X_{k+1},\ \pm X_{k+1}^2\}\cup(\pm B_{k+1}\cdot X_{k+1})] \sim \{0\}. \tag{2.1}$$

Let us suppose now that it holds for all W_j^{k+1} where $k\geqq j\geqq r+1$, and show that it holds for W_r^{k+1}.

Let P be an element of W_r^{k+1}. Then by (1.11), P is either: (a) a sum $R+S$, (b) a product $R\cdot S$, or (c) a product $b\cdot R$, where $R\in W_{r+1}^{k+1}$, $S\in W_{r+1}^s$ for some s such that $r+1\leqq s\leqq k+1$, and where $b\in B_{r+1}$.

Let us first consider case (a). By the induction hypothesis, there are integers j_1 and j_2 with $r+1\leqq j_i<k+1$ such that $R_{k+1}^*\in W_{r+1}^{j_1}$ and $S_{k+1}^*\in W_{r+1}^{j_2}$. Now clearly the monomials of $R+S$ which are not divisible by X_{k+1} are the sums of the monomials of R and S not divisible by X_{k+1}. That is, $(R+S)_{k+1}^*=R_{k+1}^*+S_{k+1}^*$. But by (1.11), $R_{k+1}^*+S_{k+1}^*\in W_r^j$ for some j such that $r\leqq j\leqq \max\{j_1,j_2\}<k+1$.

For case (b), we note that

$$P = R\cdot S = ({}^*R+R^*)\cdot({}^*S+S^*) = {}^*R\cdot{}^*S+{}^*R\cdot S^*+R^*\cdot{}^*S+R^*\cdot S^*, \tag{2.2}$$

where we have dropped the subscripts from R_{k+1}^*, etc., for compactness. Since every monomial of $^*R$ and $^*S$ is divisible by X_{k+1}, so are all those of $^*R\cdot{}^*S$, $^*R\cdot S^*$, and $R^*\cdot{}^*S$. Thus, $P_{k+1}^*=R_{k+1}^*\cdot S_{k+1}^*$, and by (1.11), $R_{k+1}^*\cdot S_{k+1}^*\in W_r^j$ for some j such that $r\leqq j\leqq\max\{j_1,j_2\}<k+1$.

To prove case (c), we note that $(b\cdot R)_{k+1}^*=b\cdot R_{k+1}^*$, and $b\cdot R_{k+1}^*$ is in $W_r^{j_1}$ by our induction hypothesis and (1.11).

LEMMA 2.2. *Let P be a nonzero element of the set W_n^m. Then P is a polynomial in X_m with coefficients in $A[X_1,\dots,X_{m-1}]$ such that $1\leqq\deg_m(P)\leqq 2^{m-n}$.*

Proof. Again we use a double induction argument. The proposition holds vacuously for W_0^0. Suppose now that it holds for all sets W_n^m where $n\leqq m\leqq k$. As the nonzero elements of W_{k+1}^{k+1} are X_{k+1} and $-X_{k+1}$, it clearly holds for this case. By (2.1), we see that it holds for W_k^{k+1}.

Suppose now that the proposition is true for all W_j^{k+1}, where $k \geqq j \geqq r+1$. We shall show that it also holds for W_r^{k+1}.

Let P be an element of W_r^{k+1}. As in Lemma 2.1, we may express P as either: (a) $P=R+S$, (b) $P=R\cdot S$, or (c) $P=b\cdot R$, where $R \in W_{r+1}^{k+1}$, $S \in W_{r+1}^s$ with $r+1 \leqq s \leqq k+1$, and $b \in B_{r+1}$. By the induction hypothesis, $1 \leqq \deg_{k+1}(R) \leqq 2^{k-r}$, and $1 \leqq \deg_{k+1}(S) \leqq 2^{k-r}$ if $s=k+1$ and $\deg_{k+1}(S)=0$ if $s<k+1$.

Now clearly P is a polynomial in X_{k+1} over $A[X_1, \ldots, X_k]$ in all these cases, since by the induction hypothesis, R and S are.

The upper bound that we must show for the degree of P in X_{k+1} is $2^{(k+1)-r}$. This is immediate for all three cases because of the induction hypothesis, and the properties of the degree of sums and products of polynomials.

To see that in case (a) the degree of P in X_{k+1} is at least one, note that if the degree of $P=R+S$ in X_{k+1} is zero, then

$$P = ({}^*R+R^*)+({}^*S+S^*) = ({}^*R+{}^*S)+(R^*+S^*) = 0+(R^*+S^*).$$

But by Lemma 2.1, there are integers j_1 and j_2 with $r+1 \leqq j_i < k+1$ such that $R_{k+1}^* \in W_{r+1}^{j_1}$ and $S_{k+1}^* \in W_{r+1}^{j_2}$. Then by (1.11), $P=R_{k+1}^*+S_{k+1}^*$ is in W_r^j for some j such that $r \leqq j \leqq \max\{j_1, j_2\} < k+1$. This is a contradiction, for $P \in W_r^{k+1}$, and by (1.11), $W_r^{k+1} \cap W_r^j = \varnothing$ if $j \neq k+1$. Thus, $\deg_{k+1}(P) \geqq 1$.

One verifies the lower bound for cases (b) and (c) in a similar manner. In case (b), if $\deg_{k+1}(P)=0$, we see from (2.2) that $P=R_{k+1}^* \cdot S_{k+1}^*$, which, by Lemma 2.1, leads again to the contradiction that $P \in W_r^j$ for some $j<k+1$.

LEMMA 2.3. *Let $\mathscr{T}((a_k), (B_k))$ be an inductive ring topology with basic neighborhoods given by* (1.18) *and* (1.19). *If $(C_k)_{k \geq 1}$ is a sequence of subsets of A such that $C_1 \subseteq C_2 \subseteq \cdots \subseteq A$ and $\bigcup_{k=1}^{\infty} C_k = A$, and if $V_n^m \cap C_m \subseteq \{0\}$ for all n and m, where $n \leqq m$, then $\mathscr{T}((a_k), (B_k))$ is Hausdorff.*

Proof. Let x be a nonzero element of A. Then there is some n such that $x \in C_m$ for all $m \geqq n$. Then $x \notin V_n^m$ for all $m \geqq n$, so $x \notin V_n = \bigcup_{m=n}^{\infty} V_n^m$. Thus (1.6) holds, so the topology is Hausdorff.

3. **Integral domains of confinality character $\aleph_0$.** Our main goal in this section is to prove (Corollary 3.2) that every countable integral domain A admits a nondiscrete, Hausdorff inductive ring topology. The countability assumption is needed only so that we may express A as a union of countably many subsets of smaller cardinality. Since domains of certain other cardinalities will also have this property, we will formulate our results in a slightly more general form.

DEFINITION. Let S be a set. Then S, or its cardinal number, has *confinality character* $\aleph_0$ if S is the union of countably many subsets of smaller cardinality([2]).

([2]) The term "cofinal with ω" is also sometimes used to define this property for cardinal numbers. We take the definition used here from [K. Gödel, *What is Cantor's continuum problem*, Amer. Math. Monthly **54** (1947), 515–525].

In what follows, the cardinality of a set S will be denoted by $|S|$. A *countable* set will always be an infinite one. We will require only the most familiar results of cardinal arithmetic, which may be found in a reference such as [1, §6, pp. 90–108].

In the next theorem, the sequence (B_k) of subsets of A can be any one satisfying conditions (1.7) and (1.8), and such that $|B_n|<|A|$ for each n.

THEOREM 3.1. *Let A be an integral domain which has confinality character $\aleph_0$. Let D be any subset of A such that $|D|=|A|$. Then there exists a sequence (a_k) of elements of D such that $\mathscr{T}((a_k),(B_k))$ is Hausdorff.*

Proof. Let (C_k) be any sequence of subsets of A which has the properties that $C_1\subseteq C_2\subseteq C_3\subseteq\cdots$, $\bigcup_{k=1}^{\infty} C_k=A$, and $|C_k|<|A|$ for each $k\geqq 1$. Our cardinality assumption on A assures the existence of such a sequence.

We will prove that we can inductively define a sequence (a_k) in D in such a way that the sets V_n^m given by (1.18) satisfy the following condition.

$$V_n^m \cap C_m \subseteq \{0\}. \tag{3.1}$$

Note that by Lemma 2.2 and (1.18), the set V_n^m depends only on the elements $a_1, a_2, \ldots, a_m$ of the sequence (a_k). Thus, we can prove (3.1) for all m and n such that $n\leqq m\leqq k$ once we have defined $a_1, a_2, \ldots, a_k$.

Since $V_0^0=\{0\}$, (3.1) holds trivially for $n=m=0$. For convenience in the proof, let a_0 be any element of D.

Assume now that $a_0, a_1, \ldots, a_k$ have been chosen from D in such a way that (3.1) holds for all m and n such that $0\leqq m\leqq n\leqq k$. We will show that there is an element a_{k+1} in D such that by taking a_{k+1} to be the next element in our defining sequence, we get that

$$V_n^{k+1} \cap C_{k+1} \subseteq \{0\} \tag{3.2}$$

for all n such that $0\leqq n\leqq k+1$.

To prove this, we first let $S_{k+1}=\{P(a_1,\ldots,a_k,X_{k+1}) : P\in W_n^{k+1}$ for some $n\leqq k+1\}$. Then the set $S'_{k+1}=S_{k+1}-C_{k+1}$ is a set of polynomials with coefficients in A in the one indeterminate X_{k+1}. Finally, let R_{k+1} be the set of all roots in A of nonzero polynomials in S'_{k+1}.

Now because $|B_{k+1}|<|A|$ and $|C_{k+1}|<|A|$, one can prove, using standard results of cardinal arithmetic, that $|S'_{k+1}|<|A|$. Since A is an integral domain, every nonzero polynomial over A has only finitely many roots. Thus, it follows that the cardinality of R_{k+1} is also less than that of A and D.

Since $|R_{k+1}|<|D|$, $D\sim R_{k+1}\neq\varnothing$, so let a_{k+1} be any element of D not in R_{k+1}. We are now able to show that with a_{k+1} chosen in this way, property (3.2) holds for all n such that $0\leqq n\leqq k+1$.

Let x be a nonzero element of V_n^{k+1}. By (1.18) and Lemma 2.2, $x=P(a_1,\ldots,a_{k+1})$

for some P in W_n^{k+1}. Let us re-express $P(a_1, \ldots, a_{k+1})$ as a polynomial in a_{k+1}. We then have

$$x = R_r(a_1, \ldots, a_k) \cdot (a_{k+1})^r + R_{r-1}(a_1, \ldots, a_k) \cdot (a_{k+1})^{r-1} + \cdots + R_0(a_1, \ldots, a_k). \tag{3.3}$$

By Lemma 2.2, $1 \leqq r \leqq 2^{k+1-n}$.

We next observe that for some $j \geqq 1$,

$$R_j(a_1, \ldots, a_k) \neq 0. \tag{3.4}$$

Suppose to the contrary that $R_j(a_1, \ldots, a_k) = 0$ for all j such that $1 \leqq j \leqq r$. Then $x = R_0(a_1, \ldots, a_k)$. But $R_0(X_1, \ldots, X_k)$ is the polynomial P_{k+1}^* of Lemma 2.1, and by that lemma, P_{k+1}^* is in W_n^j for some $j < k+1$. Then

$$x = P_{k+1}^*(a_1, \ldots, a_k) = \sigma_{(a_i)}(P_{k+1}^*(X_1, \ldots, X_k)) \in \sigma_{(a_i)}(W_n^j).$$

This is a contradiction, since x is in V_n^{k+1}, and by (1.18), $V_n^{k+1} \cap \sigma_{(a_i)}(W_n^j) = \varnothing$ for $j < k+1$.

Since (3.4) holds for some $j \geqq 1$, we may as well suppose that in (3.3),

$$R_r(a_1, \ldots, a_k) \neq 0.$$

We finally see that x is not an element of C_{k+1}, for it follows from (3.3) that

$$R_r(a_1, \ldots, a_k) \cdot (a_{k+1})^r + \cdots + R_1(a_1, \ldots, a_k) \cdot a_{k+1} + (R_0(a_1, \ldots, a_k) - x) = 0. \tag{3.5}$$

Now

$$R_r(a_1, \ldots, a_k) \cdot (X_{k+1})^r + \cdots + R_1(a_1, \ldots, a_k) \cdot X_{k+1} + R_0(a_1, \ldots, a_k)$$

is in S_{k+1}, so if x is in C_{k+1}, then (3.4) and (3.5) show that a_{k+1} is the root of a nonzero polynomial in S'_{k+1}. This is a contradiction, since a_{k+1} is not in R_{k+1}. Thus, $x \notin C_{k+1}$, and we have verified that (3.2) holds.

This completes the inductive step. We have shown, then, that we can define a sequence (a_k) such that (3.1) holds. It follows from Lemma 2.3 that $\mathscr{T}((a_k), (B_k))$ is Hausdorff.

Since any countable integral domain clearly has confinality character $\aleph_0$, the following corollary is an immediate consequence of the theorem.

COROLLARY 3.2. *If A is a countable integral domain and (b_k) is any sequence of distinct elements of A, then for some subsequence (a_k) of (b_k), $\mathscr{T}((a_k), (B_k))$ is Hausdorff.*

REMARK 3.3. Notice in the proof of Theorem 3.1 that we did not make use of the set C_{k+1} in inductively defining the sequence (a_n) until we came to defining a_{k+1}. Thus, we do not have to assume that the sequence (C_n) is given to us from the beginning, but may at the $(k+1)$th stage, take C_{k+1} to be in some way dependent on the particular choices of $a_1, a_2, \ldots, a_k$. It will be necessary to do this in several of our applications.

4. **Algebraically unbounded rings.** We turn our attention in this section to proving the existence of Hausdorff inductive topologies on rings which are described by the following definition.

DEFINITION. A commutative ring A with an identity element, 1, is *algebraically unbounded* if there is a sequence $(B_k)_{k\geqq 1}$ of subrings of A such that $1\in B_1\subseteq B_2 \subseteq B_3\subseteq \cdots$, $A=\bigcup_{k=1}^{\infty} B_k$, and such that for all pairs of positive integers (n, m), there is an element a of A of degree at least m over the subring B_n. The sequence (B_k) will be called an *algebraically unbounded sequence* of subrings of A.

By the *degree over* B_n of an element a of A, we mean the least integer r such that a is a root of a polynomial over B_n of degree r. If a is a root of no nonzero polynomial over B_n, i.e., a is *transcendental* over B_n, then we will say that it has *infinite degree* over B_n. Its degree is then greater than m for every positive integer m.

Suppose now that (B_k) is an algebraically unbounded sequence of subrings of A, and suppose that we use the subrings B_k to define an inductive polynomial topology. Since the sets W_n^m are formed by operations of addition and multiplication, from (1.10) and (1.11), we can prove inductively that W_n^m is contained in $B_m[X_1, \ldots, X_m]$ for $n=m, m-1, \ldots, 0$. It will follow, then, that no matter how a sequence (a_k) is chosen, the sets V_n^m determined by it will be contained in the subrings B_k. Indeed, if r is an integer at least m such that all the elements $a_1, a_2, \ldots, a_m$ are contained in B_r, then

$$V_n^m \subseteq \sigma_{(a_k)}(W_n^m) \subseteq \sigma_{(a_k)}(B_m[X_1, \ldots, X_m]) = B_m[a_1, \ldots, a_m] \subseteq B_r,$$

for all n such that $0\leqq n\leqq m$. This condition on the sets V_n^m will be instrumental in the proof of our next theorem.

THEOREM 4.1. *Let A be an algebraically unbounded commutative ring with identity. Then there are Hausdorff inductive ring topologies on A.*

Proof. Let $(B_k)_{k\geqq 1}$ be an algebraically unbounded sequence of subrings of A. We will inductively choose a sequence (a_k) such that the sets V_n^m for $\mathscr{T}((a_k), (B_k))$ satisfy the following condition.

$$V_n^m \cap B_m \subseteq \{0\}. \tag{4.1}$$

To begin the sequence conveniently, we will augment it by letting a_0 be any element of A. Assume now that $a_0, \ldots, a_k$ have been defined. Let $\rho(k)$ be the least integer r such that $\{a_0, \ldots, a_k\}\subseteq B_r$. Then take a_{k+1} to be any element of A whose degree over $B_{\rho(k)}$ is greater than 2^{k+1}. By our hypothesis on A, such an a_{k+1} exists.

We first note that the sequence $(\rho(k))_{k\geqq 0}$ which was just defined is strictly increasing. Thus, since $\rho(0)$ is at least one, by induction, $\rho(k)\geqq k+1$ for all $k\geqq 0$. We have then that for all $k\geqq 0$, $B_{k+1}\subseteq B_{\rho(k)}$.

To see that (4.1) holds for the sets V_n^m determined by the sequence (a_k) which we have chosen, let x be any nonzero element of V_n^m. Then by Lemma 2.2 and (1.18), we may express x as $x=P(a_1, a_2, \ldots, a_m)$, where $P \in W_n^m$. As was observed above, $P \in B_m[X_1, \ldots, X_m]$. If we re-express P as a polynomial in X_m, we have

$$(4.2) \quad x = R_k(a_1, \ldots, a_{m-1}) \cdot (a_m)^k + \cdots + R_1(a_1, \ldots, a_{m-1}) \cdot a_m + R_0(a_1, \ldots, a_{m-1}),$$

where we assume that $R_k(a_1, \ldots, a_{m-1}) \neq 0$. If $k=0$, then, as in the proof of Theorem 3.1, we have by Lemma 2.1 that x is in $\sigma_{(a_j)}(W_n^i)$ for some i such that $n \leqq i \leqq m-1$, which is a contradiction. Thus, we have that $k \geqq 1$, and by Lemma 2.2, $k \leqq 2^{m-n} \leqq 2^m$.

Now $B_m \subseteq B_{\rho(m-1)}$ and P is in $B_m[X_1, \ldots, X_m] \subseteq B_{\rho(m-1)}[X_1, \ldots, X_m]$. Thus, as the set $\{a_1, \ldots, a_{m-1}\}$ is contained in $B_{\rho(m-1)}$, the coefficients $R_j(a_1, \ldots, a_{m-1})$ of (4.2) are also in $B_{\rho(m-1)}$.

Now suppose that x is in B_m. Then x is in $B_{\rho(m-1)}$, so by (4.2), a_m is the root of a nonzero polynomial, of degree at most 2^m, with coefficients in $B_{\rho(m-1)}$. This contradicts the fact that we chose a_m to be of degree greater than 2^m over $B_{\rho(m-1)}$. Thus, we must conclude that x is not in B_m, and so (4.1) is proven.

Again, it follows from (4.1) and Lemma 2.3 that the topology $\mathscr{T}((a_k), (B_k))$ is Hausdorff.

5. **Inductive topologies on fields.** We will show in this section how the results from §§3 and 4 can be used to prove the existence of nondiscrete, Hausdorff field topologies on every infinite field.

Let F be a subfield of K. It is well known [9, §12, pp. 95–102] that there is a subset D of K algebraically independent over F such that K is an algebraic extension of $F(D)$. Such a set D is called a *transcendence basis*, and $F(D)$ can be regarded as a field of rational functions, with the elements of D regarded as indeterminates.

LEMMA 5.1. *Let F be a subfield of a field K, and let D be a transcendence basis for K over F. If D is infinite, then K is algebraically unbounded.*

Proof. Let (D_k) be a sequence of subsets of D such that $D_1 \subseteq D_2 \subseteq \cdots$, $\bigcup_{k=1}^{\infty} D_k = D$, and $D \sim D_k \neq \varnothing$ for all $k \geqq 1$. For each k, let F_k be the algebraic closure of $F(D_k)$ in K. We will show that $(F_k)_{k \geqq 1}$ is an algebraically unbounded sequence of subrings of K.

First, it is clear that $F_1 \subseteq F_2 \subseteq \cdots$. Second, if $a \in K$, then there is a polynomial $P = \sum_{i=0}^{n} \alpha_i \cdot X^i$ with coefficients in $F(D)$ such that a is a root of P. Since the α_i's are rational expressions in the elements of D, each involves only finitely many elements of D. Thus, each α_i is in $F(D_{m(i)})$ for some $m(i)$. If $m \geqq m(i)$ for $1 \leqq i \leqq n$, then all the coefficients of P are in $F(D_m)$. Hence, a is in F_m, and it follows that $\bigcup_{k=1}^{\infty} F_k = K$.

To see that K contains elements of arbitrarily high degree over any one of the subfields F_n, consider an element x of $D \sim D_n$. If x were algebraic over F_n, then it would also be algebraic over $F(D_n)$, [9, Theorem C, p. 61], which would violate the algebraic independence of D over F. Thus, x is transcendental over F_n.

THEOREM 5.2. *Every infinite field K admits a nondiscrete, Hausdorff inductive ring topology.*

Proof. Let F be the prime subfield of K, and let D be a transcendence basis for K over F.

Case 1. D is finite. Then K is a countable field, so the result follows from Corollary 3.2.

Case 2. D is infinite. Then by Lemma 5.1, K is algebraically unbounded, so the result follows from Theorem 4.1.

COROLLARY 5.3. *Every infinite field admits a nondiscrete, Hausdorff field topology*

Proof. It is known [4, pp. 809–811] that if $\mathscr{T}$ is a nondiscrete, Hausdorff ring topology on a field K, then there is a nondiscrete, Hausdorff field topology $\mathscr{T}'$ on K coarser than $\mathscr{T}$. The desired result follows from this fact, and the theorem.

6. **Locally bounded topologies on fields.** The topologies given to fields in the preceding section do not necessarily have the desirable property of local boundedness. We investigate here the question of which fields admit topologies with this additional property. Although local boundedness can be built into inductive topologies (see §8), the methods of valuation theory will be more efficacious here than our inductive technique.

DEFINITION. If $\mathscr{T}$ is a ring topology on a commutative ring A and $\mathscr{U}$ is a basic system of neighborhoods of zero, then a subset B of A is *bounded* if for all U in $\mathscr{U}$, there is a V in $\mathscr{U}$ such that $B\cdot V\subseteq U$. If there is a bounded neighborhood of zero, then $\mathscr{T}$ is *locally bounded.*

DEFINITION. If K is a field, a *real valuation* on K is a function ϕ from $K^*=K\sim\{0\}$ into the positive real numbers such that for all x and y in K^*, the following properties hold.

$$\phi(xy) = \phi(x)\cdot\phi(y), \quad \phi(x+y) \leqq \max\{\phi(x), \phi(y)\}.$$

We will call ϕ *proper* if its range contains more than one element.

One gets a locally bounded field topology from a valuation ϕ by taking the sets of the form

$$U_\varepsilon = \{x \in K^* : \phi(x) \leqq \varepsilon\} \cup \{0\}$$

for every $\varepsilon>0$ as a basic system of neighborhoods of zero. The topology is Hausdorff, and is nondiscrete if and only if ϕ is proper.

DEFINITION. We will say that a field is *algebraic* if it is of prime characteristic, and is algebraic over its prime subfield.

THEOREM 6.1. *The following are equivalent for a field K.*

1° *K is not algebraic.*

2° *K admits a proper, real valuation.*

3° *K admits a nondiscrete, Hausdorff, locally bounded ring topology.*

Proof. The fact that 2° implies 3° was noted above.

3° implies 1°. Suppose that K is algebraic and that $\mathscr{T}$ is a locally bounded, nondiscrete ring topology on K. We will show that every neighborhood U of zero contains all of K, and hence, that $\mathscr{T}$ is the indiscrete topology and therefore not Hausdorff.

Since $\mathscr{T}$ is locally bounded, a neighborhood U of zero contains a neighborhood V of zero such that $V \cdot V \subseteq V$. (Let $V = \{x \in U_0 : xU_0 \subseteq U_0\}$, where U_0 is a bounded neighborhood of zero contained in U.) Then for every positive n, $V^n \subseteq V$.

Now since K is algebraic, for any nonzero x, $x^{-1} = x^n$ for some positive integer n. Thus, if $x \in V$, then $x^{-1} = x^n \in V^n \subseteq V$, so $V^{-1} \subseteq V$.

Now let x be any element of K. Then there is a neighborhood W of zero such that $xW \subseteq V$. As $\mathscr{T}$ is nondiscrete, there is a nonzero element y in $W \cap V$. Then $xy \in V$, so $x \in Vy^{-1} \subseteq VV^{-1} \subseteq V \subseteq U$. Thus, $K \subseteq V \subseteq U$, so $\mathscr{T}$ is the indiscrete topology.

1° implies 2°. If K is not algebraic, then either K has characteristic zero, or contains an element τ transcendental over its finite prime subfield Z_p. In these respective cases, let F be the subfield of rational numbers, or $Z_p(\tau)$. Then in either case, F clearly admits a proper, real valuation ϕ. Our goal is to extend ϕ to K.

Let B be a transcendence basis for K over F. Then the elements of $F' = F(B)$ are quotients P/Q of polynomial expressions in the elements of B with coefficients in F. For such a polynomial $P = \sum_{i=1}^{n} a_i \cdot X_{\alpha_1}^{m_{i,1}} \cdots X_{\alpha_k}^{m_{i,k}}$, where $X_{\alpha_i} \in B$ and $a_i \in F$, we define $\phi(P)$ by

$$\begin{aligned} \phi(P) &= 0 && \text{if } a_i = 0 \text{ for all } i, \\ &= \max\{\phi(a_i) : 1 \leqq i \leqq n\} && \text{if } a_i \neq 0 \text{ for some } i. \end{aligned}$$

We then define $\phi(P/Q) = \phi(P)/\phi(Q)$. One can show that these definitions extend ϕ to a valuation on F'. Since K is an algebraic extension of F', it is known [8, Theorem 12, p. 57] that ϕ can be extended to a valuation on K.

7. Other applications of inductive topologies.

In this section we will present generalizations of some of the results proven by Hinrichs for the integers, and will show how his results can be derived within our more general context. Theorem 7.2 is a heretofore unpublished result of Hinrichs'.

THEOREM 7.1 [6, p. 993]. *There exist Hausdorff ring topologies on the ring of integers, Z, which have neighborhoods of zero which do not contain ideals.*

Proof. Let $B_n = \{m \in Z : 0 \leqq m \leqq n\}$ for every $n \geqq 1$. By Corollary 3.2, there is a sequence (a_k) of positive integers such that $\mathscr{T}((a_k), (B_k))$ is Hausdorff. We will show that the sequence (a_k) can be chosen so that the neighborhood V_0 of zero for this topology, given by (1.19), has the property that for all positive integers m,

$$\text{there is an interval } I_m = [k_m, k_m + m] \text{ in } Z \text{ such that } V_0 \cap I_m = \varnothing. \tag{7.1}$$

This will assure that for all $m \geqq 1$, V_0 does not contain the ideal mZ.

To get (7.1) to hold, we make use of Remark 3.3 in defining the sequence (a_k). We may let a_1 be any element of Z. Assume now that $a_1, \ldots, a_m$ have been defined. By Remark 3.3, and the fact that the sets V_n^i are in this case finite, we may take the set C_{m+1} of (3.1) to contain $I_m = [k_m, k_m+m]$, where $k_m = \sup(\bigcup_{i=0}^m V_0^i)+1$. But then for all $k \geqq 0$, $V_0^k \cap I_m = \varnothing$. This follows from the definition of I_m for $k \leqq m$, and from (3.1) for $k \geqq m+1$, since $I_m \subseteq C_k$ for all such k. Thus, (7.1) and the theorem follow.

Definition. A ring topology is said to be *additively generated* if the additive subgroup generated by any neighborhood of zero is the entire ring.

One can readily see that if the elements in a sequence (a_k) of integers are chosen so that for all m there is an $n>m$ such that a_m and a_n are relatively prime, then $\mathscr{T}((a_k), (B_k))$ is additively generated. We thus have that *there are Hausdorff, additively generated ring topologies on the integers.*

Correl proved [3, Theorem 2.10, p. 38] that a ring topology on the rational numbers, Q, which is not additively generated is finer than the p-adic topology for some prime p. This might lead one to wonder if additively generated topologies are necessarily finer than the usual one. The following theorem shows that this is not the case.

Theorem 7.2 [Hinrichs]. *There are additively generated, Hausdorff ring topologies on Q not finer than the usual one.*

Proof. By Corollary 3.2, there is a Hausdorff inductive ring topology $\mathscr{T}_0 = \mathscr{T}((a_k), (B_k))$ on Q, where (a_k) is a subsequence of the sequence of prime integers. (The sequence (B_k) can be any increasing sequence of finite subsets of Q whose union is Q.)

To show that $\mathscr{T}_0$ is additively generated, let G be any additive subgroup neighborhood of zero, and let a/b be any element of Q. Then $b \neq 0$, so bG is again an additive subgroup neighborhood of zero. Hence, bG contains all but finitely many of the primes a_k. Thus, $bG \cap Z$ is a subgroup, and hence an ideal in Z which contains infinitely many primes. It follows that $bG \cap Z = Z$, so $a \in bG$, and hence, $a/b \in G$.

We see, then, that $G = Q$, so $\mathscr{T}_0$ is additively generated. Clearly $\mathscr{T}_0$ is not finer than the usual topology, for the sequence (a_k) which converges to zero in $\mathscr{T}_0$ is bounded away from zero in the usual topology.

Using Correl's theorem and taking (a_k) to be a subsequence of the sequence $(1/p_k)$ of inverses of primes, it can be shown that there also are additively generated ring topologies on Q strictly finer than the usual one.

In our next two theorems, we show that not only does the inductive approach give us access to Hausdorff topologies on the rings in the classes considered in §3 and §4, but it gives a way of proving that there are uncountably many of them. These generalize Hinrichs' result [6, p. 94] for the integers.

Lemma 7.3. *Let A be a ring, and let $\mathscr{T}$ be a ring topology on A in which a sequence*

(a_k) *of distinct, cancellable elements converges to zero. If U is any $\mathcal{T}$-open set, then* $|U|=|A|$.

Proof. It is clearly sufficient to take U to be a neighborhood of zero.

For all x in A, $xa_n \in U$ for some n. Thus, $A=\bigcup_{n=1}^{\infty} A_n$, where $A_n=\{x \in A : xa_n \in U\}$. Since the a_n's are cancellable, the mappings $x \to xa_n$ are injections on A. Thus $|A_n| \leqq |U|$ for each n. Since U is infinite, $|A|=|\bigcup_{n=1}^{\infty} A_n| \leqq |U|$. Since $U \subseteq A$, $|U|=|A|$.

THEOREM 7.4. *Let A be an integral domain of confinality character $\aleph_0$. Then there are uncountably many first countable, Hausdorff ring topologies on A.*

Proof. Suppose to the contrary that all the nondiscrete, first countable, Hausdorff ring topologies on A can be enumerated, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$. For each $n \geqq 1$, let $\mathcal{U}_n=\{U_k(n) : k \geqq 1\}$ be a countable basis for the neighborhoods of zero for $\mathcal{T}_n$ such that $U_{k+1}(n) \subseteq U_k(n)$ for each $k \geqq 1$. Using the usual diagonal process, Theorem 3.1, and Remark 3.3, we will obtain a Hausdorff inductive topology $\mathcal{T}_0$, and for each $n \geqq 1$, a sequence $(b_k(n))_{k \geqq 1}$ convergent to zero in $\mathcal{T}_n$ but bounded away from zero in $\mathcal{T}_0$. This will imply that $\mathcal{T}_0$ is not in the list, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$, which is a contradiction.

The sequence of sets B_k which determines $\mathcal{T}_0$ can be any one satisfying (1.7) and (1.8) and such that $|B_k|<|A|$ for all $k \geqq 1$. The sequences $(b_k(n))_{k \geqq 1}$ will be kept away from zero by including elements from them in the sets C_k of (3.1). To get the C_k's large enough, however, we begin with a sequence (C_k') such that $C_1' \subseteq C_2' \subseteq \cdots$, $\bigcup_{k=1}^{\infty} C_k'=A$, and $|C_k'|<|A|$.

We now describe how the sequence (a_k) determining $\mathcal{T}_0$, and the sequences $(b_k(n))$ for $n \geqq 1$ and (C_k) are to be defined.

Let $C_1=C_1'$, and let a_1 be chosen according to the procedure in Theorem 3.1 so that (3.1) holds, i.e., $V_i^1 \cap C_1 \subseteq \{0\}$ for $i=0, 1$. Now, as was observed in proving Theorem 3.1, $|V_0^0 \cup V_0^1|<|A|$. Since $|U_1(1)|=|A|$ by Lemma 7.3, we can find an element $b_1(1)$ in $U_1(1) \sim [V_0^0 \cup V_0^1]$.

Assume now that $a_1, \ldots, a_k$; $C_1, \ldots, C_k$ have been defined so that (3.1) holds for $n \leqq m \leqq k$. Also assume that partial sequences

$$\begin{array}{l} b_1(1), b_2(1), \ldots, b_{k-1}(1), b_k(1) \\ b_1(2), b_2(2), \ldots, b_{k-1}(2) \\ \quad\vdots \qquad \vdots \\ b_1(k-1), b_2(k-1) \\ b_1(k) \end{array}$$

have been defined so that

$$b_i(j) \in U_i(j) \sim \bigcup_{m=0}^{i+j-1} V_0^m \quad \text{for } 2 \leqq i+j \leqq k+1. \tag{7.2}$$

We then let

$$(7.3) \qquad C_{k+1} = C'_{k+1} \cup \{b_i(j) : 2 \leqq i+j \leqq k+1\},$$

which we may do by Remark 3.3. Clearly we still have $|C_{k+1}| < |A|$. Next choose a_{k+1} so that (3.1) holds for $n \leqq m \leqq k+1$. Finally, since $|\bigcup_{m=0}^{k+1} V_0^m| < |A|$, we may by Lemma 7.3 pick $k+1$ elements $b_1(k+1), b_2(k), \ldots, b_k(2), b_{k+1}(1)$ such that for $1 \leqq i \leqq k+1$,

$$b_i(k+2-i) \in U_i(k+2-i) \sim \bigcup_{m=0}^{k+1} V_0^m.$$

Now the topology $\mathscr{T}_0 = \mathscr{T}((a_k), (B_k))$ just defined is Hausdorff by Theorem 3.1. One can see from (7.2) that the sequence $(b_k(n))_{k \geqq 1}$ converges to zero in $\mathscr{T}_n$. It follows from (7.2), (7.3), and (3.1) that $b_i(j) \notin V_0$ for all i and j, so that each of the sequences $(b_k(n))_{k \geqq 1}$ is bounded away from zero in $\mathscr{T}_0$.

This contradiction leads us to conclude that there are uncountably many first countable, Hausdorff ring topologies on A.

Our next theorem gives the result analogous to Theorem 7.4 for algebraically unbounded rings. We will only sketch the proof.

THEOREM 7.5. *Let $(B_k)_{k \geqq 1}$ be an algebraically unbounded sequence of subrings of a commutative ring A with identity. Then there are uncountably many Hausdorff inductive ring topologies on A determined by the sequence (B_k).*

Proof. Assume that $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \ldots$ is a list of all of the Hausdorff inductive topologies determined by the sequence (B_k). Let $(b_k(n))_{k \geqq 1}$ be a sequence convergent to zero in $\mathscr{T}_n$ for each $n \geqq 1$.

One can use a diagonal process just as in the proof of Theorem 7.4 to get subsequences of each of the sequences $(b_k(n))$ which are bounded away from zero in a Hausdorff inductive topology $\mathscr{T}_0$. The facts making this possible are that there are elements from each of the sequences $(b_k(n))$ in $B_m \sim B_{m-1}$ for arbitrarily large integers m, and that each of the sets V_0^m for the topology $\mathscr{T}_0$ is, by Theorem 4.1, contained in one of the subrings B_r.

In our final theorem of this section, we prove that on all of the rings we have considered, there are ring topologies which are not first countable, thereby generalizing another result of Hinrichs' [6, p. 995].

THEOREM 7.6. *If A is an integral domain of confinality character $\aleph_0$ or an algebraically unbounded commutative ring with identity, then there are Hausdorff ring topologies on A which are not first countable.*

Proof. Let (B_k) be an increasing sequence of subsets of A such that $|B_k| < |A|$ for all k and $\bigcup_{k=1}^{\infty} B_k = A$, or an algebraically unbounded sequence of subrings for these respective cases. Let $\mathscr{M}$ be a maximal chain in the nonempty collection, ordered by set inclusion, of all nondiscrete, Hausdorff inductive topologies on A determined by the sequence (B_k). Let $\mathscr{T}$ be the supremum of $\mathscr{M}$.

Then $\mathscr{T}$ is a nondiscrete, Hausdorff ring topology on A in which the sets B_k are bounded. If $\mathscr{T}$ were first countable, then one could get a basis $\mathscr{V}=\{V_n : n\geqq 1\}$ for the neighborhoods of zero satisfying (1.1)–(1.4) and (1.5′). One can see, then, that if $a_k \in V_k \sim \{0\}$ for each $k\leqq 1$, then $\mathscr{T}((a_k), (B_k))$ is a nondiscrete topology finer than $\mathscr{T}$. Furthermore, using the technique of the proof of Theorem 7.4, one can keep a sequence convergent to zero in $\mathscr{T}$ bounded away from zero in $\mathscr{T}((a_k), (B_k))$, and thus assure that the latter topology is strictly finer than $\mathscr{T}$.

This clearly violates the maximality of the chain $\mathscr{M}$, so we must conclude that $\mathscr{T}$ is not first countable.

8. **Characterization results.** Two properties shared by all inductive ring topologies are the properties of first countability and countable boundedness. We say that a ring topology on a ring A is *countably bounded* if A is the union of countably many sets bounded with respect to the topology.

To see that a topology $\mathscr{T}((a_k), (B_k))$ on a ring A has this latter property, note that by (1.5′) the sets B_k are bounded. The property follows, then, from the fact that

$$A = \bigcup_{k=1}^{\infty} (B_1\cdot B_2\cdots B_k \cup -B_1B_2\cdots B_k),$$

and that products and finite unions of bounded sets are bounded.

It is natural to wonder if these two properties characterize all ring topologies which can be defined inductively. We shall see that they at least characterize for fields, the "weak" inductive topologies in the following sense.

DEFINITION. A *weak inductive ring topology* is a topology $\mathscr{T}((a_k), (B_k))$ derived in exactly the same manner as an inductive ring topology, except that instead of (1.7) and (1.8), the sets B_k satisfy the following conditions.

$$B_k = B_k' \cup B_k'' \quad \text{for all } k \geqq 1. \tag{8.1}$$

$$B_1' \subseteq B_2' \subseteq B_3' \subseteq \cdots. \tag{1.7′}$$

$$D' \cup -D' \text{ multiplicatively generates } A,\ D' = \bigcup_{k=1}^{\infty} B_k'. \tag{1.8′}$$

THEOREM 8.1. *Let $\mathscr{T}$ be a ring topology on a commutative ring A with identity such that zero is a limit point of the set $S=\{a \in A$: The map $x \to ax$ is open$\}$. Then $\mathscr{T}$ is a weak inductive ring topology if and only if $\mathscr{T}$ is first countable and countably bounded.*

Proof. The necessity of these two conditions holding for an inductive topology was just noted. The observation is equally valid for weak inductive topologies.

Sufficiency. Let (C_k) be a sequence of bounded subsets such that $A=\bigcup_{k=1}^{\infty} C_k$. For each n, let $B_n'=\bigcup_{k=1}^{n} C_k$. Then the sets B_n' are bounded, and satisfy (1.7′) and (1.8′).

Since $\mathscr{T}$ is first countable, we clearly can get a basic system $\{U_n : n \geqq 0\}$ of neighborhoods of zero which satisfies the following conditions for all $n \geqq 0$.

$$U_n = -U_n, \tag{8.2}$$

$$U_{n+1} + U_{n+1} \subseteq U_n, \tag{8.3}$$

$$U_{n+1} \cdot U_{n+1} \subseteq U_n, \tag{8.4}$$

$$B'_{n+1} \cdot U_{n+1} \subseteq U_n. \tag{8.5}$$

Now let $B_n = B'_n \cup U_n$, and let a_n be any element of $U_n \cap S$ for each $n \geqq 1$. We will show that $\mathscr{T}$ is the weak inductive topology $\mathscr{T}_0 = \mathscr{T}((a_k), (B_k))$.

Since $a_n \in U_n$ for each n, it is an easy inductive proof to show that each of the sets V_n^m given by (1.18) for $\mathscr{T}_0$ is contained in U_n. Thus, the $\mathscr{T}_0$-neighborhood of zero V_n is contained in U_n for each n, so $\mathscr{T} \subseteq \mathscr{T}_0$.

Since $U_{n+1} \subseteq B_{n+1}$, clearly $U_{n+1} \cdot a_{n+1} \subseteq V_n^{n+1} \subseteq V_n$, by (1.11) and (1.18). Since $a_{n+1} \in S$, $a_{n+1} \cdot U_{n+1}$ is a $\mathscr{T}$-neighborhood of zero, so $\mathscr{T}_0 \subseteq \mathscr{T}$.

COROLLARY 8.2. *A ring topology $\mathscr{T}$ on a field is a weak inductive ring topology if and only if it is first countable and countably bounded.*

We have not been able to prove that every first countable, countably bounded ring topology on an arbitrary commutative ring with identity is inductive. However, the following theorem asserts that all such topologies can be in a sense approximated by inductive topologies.

THEOREM 8.3. *If $\mathscr{T}$ is a first countable, countably bounded ring topology on a commutative ring A with identity, then $\mathscr{T}$ is the infimum (in the lattice of all topologies on A) of the set of all inductive ring topologies which are finer than $\mathscr{T}$.*

Proof. Let $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ be a chain of bounded subsets of A such that $A = \bigcup_{n=1}^{\infty} B_n$. As in Theorem 8.1, one can inductively define a basic system $\{U_n : n \geqq 0\}$ of neighborhoods of zero for $\mathscr{T}$ satisfying conditions (8.2)–(8.4) for all $n \geqq 0$, and the further condition that $B_{n+1} \cdot U_{n+1} \subseteq U_n$.

As was observed, for any sequence (a_k) such that $a_k \in U_k$ for all k, the inductive ring topology $\mathscr{T}((a_k), (B_k))$ is finer than $\mathscr{T}$. Thus, we see that the set $\mathscr{A}$ of inductive ring topologies finer than $\mathscr{T}$ is nonempty.

Let $\mathscr{T}_0 = \inf \mathscr{A}$. Then clearly $\mathscr{T}_0$ is a finer topology than $\mathscr{T}$, i.e., $\mathscr{T} \subseteq \mathscr{T}_0$. Suppose that $\mathscr{T}_0$ is strictly finer than $\mathscr{T}$. This is equivalent to saying that the identity map I from $(A, \mathscr{T})$ onto $(A, \mathscr{T}_0)$ is not continuous. That is, I does not preserve $\mathscr{T}$-limit points.

Thus, there is a subset S of A, and a point x in A such that x is a $\mathscr{T}$-limit point of S but not a $\mathscr{T}_0$-limit point of S. As $\mathscr{T}$ is first countable, we may extract a sequence (x_n) from S such that (x_n) converges to x in $\mathscr{T}$.

As (x_n) converges to x in $\mathscr{T}$, $(x_n - x)$ converges to zero in $\mathscr{T}$. Let us choose a subsequence (x_{n_k}) of (x_n) so that for each k, $x_{n_k} - x \in U_k$.

Then, as was noted above, $\mathcal{T}((x_{n_k}-x), (B_k))$ is an inductive ring topology finer than $\mathcal{T}$, and hence, $\mathcal{T}_0 \subseteq \mathcal{T}((x_{n_k}-x), (B_k))$. But in $\mathcal{T}((x_{n_k}-x), (B_k))$, the sequence $(x_{n_k}-x)$ converges to zero, and hence, (x_{n_k}) converges to x. Now as $\mathcal{T}_0$ is coarser than $\mathcal{T}((x_{n_k}-x), (B_k))$, it follows that (x_{n_k}) also converges to x in $\mathcal{T}_0$. This is a contradiction, since $\{x_{n_k} : k \geqq 1\} \subseteq S$, and x is not a $\mathcal{T}_0$-limit point of S. Hence, we conclude that $\mathcal{T} = \mathcal{T}_0$.

As has been observed, inductive topologies do not in general have the property of local boundedness. (An example of one which does not is gotten by taking a nondiscrete, Hausdorff inductive topology on an algebraic field. By Theorem 6.1, this topology is not locally bounded.) However, local boundedness is a sufficient condition for a first countable ring topology on a field to be inductive, as we shall see shortly. As a corollary of this result, we will get an interesting characterization of all first countable, locally bounded ring topologies on a countable field. To do this, however, we will need a way of building local boundedness into an inductive topology, which the following definition gives us.

Definition. An *inductive locally bounded* ring topology on A, denoted by $\mathcal{T}_b((a_k), (B_k))$ is derived in exactly the same manner as an inductive ring topology, except that the definition of the set W_r^{k+1} for $r \leqq k$ is changed from (1.11) to the following.

$$W_r^{k+1} = \left[\left(W_{r+1}^{k+1} + \bigcup_{s=r+1}^{k+1} W_{r+1}^s\right) \cup \left(W_{r+1}^{k+1} \cdot \bigcup_{s=r+1}^{k+1} W_{r+1}^s\right) \cup (B_{r+1} \cdot W_{r+1}^{k+1}) \cup \left(\left(\bigcup_{s=0}^{k} W_0^s\right) \cdot \left(\bigcup_{s=r+1}^{k+1} W_{r+1}^s\right)\right)\right] \sim \left[\bigcup_{s=r}^{k} W_r^s\right]. \tag{8.6}$$

This change clearly builds into the set W_0 the property that $W_0 \cdot W_{n+1} \subseteq W_n$ for all $n \geqq 0$. Then also by (1.19), $V_0 \cdot V_{n+1} \subseteq V_n$ for all n, so V_0 is a bounded neighborhood of zero.

With local boundedness built into an inductive topology in this way, the problem of finding sufficient conditions for Hausdorffness becomes very difficult, since Lemmas 2.1 and 2.2 are no longer true.

Theorem 8.4. *Let K be a field, and let $\mathcal{T}$ be a first countable, locally bounded ring topology on K. Then $\mathcal{T}$ is both an inductive ring topology and an inductive locally bounded ring topology.*

Proof. Since the discrete topology is $\mathcal{T}((a_k), (B_k))$ or $\mathcal{T}_b((a_k), (B_k))$, where $a_k = 0$ for all k, and therefore is both inductive and inductive locally bounded, let us assume in what follows that $\mathcal{T}$ is nondiscrete.

Let U be a bounded, symmetric neighborhood of zero such that $1 \in U$ and $U \cdot U \subseteq U$. Let $(b_n)_{n \geqq 1}$ be any sequence of nonzero elements which converges to zero. Then $\{b_n U : n \geqq 1\}$ is a basic system of neighborhoods of zero for $\mathcal{T}$.

We will inductively extract from (b_n) a subsequence $(a_k)=(b_{n_k})$ such that the basic system $\{a_kU : k\geqq 1\}$ has certain special properties.

To begin the inductive definition, let $a_0=1$. Assume now that $a_0, a_1=b_{n_1}$, $a_2=b_{n_2}, \ldots, a_k=b_{n_k}$ have been defined. As a_kU is a neighborhood of zero, we can find neighborhoods N_1-N_4 of zero such that

$$N_1+N_1 \subseteq a_kU,$$

$$N_2\cdot N_2 \subseteq a_kU,$$

$$T_k\cdot N_3 \subseteq a_kU,$$

$$N_4\cdot U \subseteq a_kU.$$

Here, and in what follows, T_k is the finite set $\{(a_{r-1}/a_r) : 1\leqq r\leqq k\}$ for all $k\geqq 1$.

Now fix some integer $r>n_k$ such that $b_rU\subseteq\bigcap_{j=1}^4 N_j$. Let $a_{k+1}=b_{n_{k+1}}=b_r$.

Letting $U_n=a_nU$ for each $n\geqq 0$, we then have that $\{U_n : n\geqq 0\}$ is a basic system of $\mathcal{T}$-neighborhoods of zero which satisfies the following conditions for all $n\geqq 0$.

(8.7) $$U_{n+1}+U_{n+1} \subseteq U_n,$$

(8.8) $$U_{n+1}\cdot U_{n+1} \subseteq U_n,$$

(8.9) $$T_{n+1}\cdot U_{n+1} \subseteq U_n,$$

(8.10) $$U_{n+1}\cdot U_0 \subseteq U_n.$$

We next note that $D'=U\cup\{(a_n/a_{n+1}) : n\geqq 0\}$ multiplicatively generates A. To see this, let x be any element of A. As (a_k) converges to zero, a_nx is in U for some n. Since

$$x = (xa_n)(a_{n-1}/a_n)(a_{n-2}/a_{n-1})\cdots(a_1/a_2)(a_0/a_1),$$

we see that x is a product of elements of D'.

Let (S_k) be a sequence of subsets of U such that $S_1\subseteq S_2\subseteq\cdots\subseteq U$, and $\bigcup_{k=1}^\infty S_k=U$. For each $k\geqq 1$, let

(8.11) $$B_k = S_k \cup T_k\cdot S_k \cup T_k.$$

Then $B_1\subseteq B_2\subseteq\cdots$, and since $D'\subseteq D=\bigcup_{k=1}^\infty B_k$, D multiplicatively generates A. Also, one may easily see that $B_{n+1}\cdot U_{n+1}\subseteq U_n$ for all $n\geqq 0$. As a_k is in U_k for $k\geqq 1$, it follows, as in Theorem 8.1, that the inductive ring topology $\mathcal{T}_0=\mathcal{T}((a_k),(B_k))$ is finer than $\mathcal{T}$. That is, $\mathcal{T}\subseteq\mathcal{T}_0$. Similarly, the inductive locally bounded topology $\mathcal{T}_1=\mathcal{T}_b((a_k),(B_k))$ is also finer than $\mathcal{T}$.

We shall now show that $\mathcal{T}_0\subseteq\mathcal{T}$ and $\mathcal{T}_1\subseteq\mathcal{T}$. Considering both cases together, since one argument suffices for both, let $\{V_n : n\geqq 0\}$ be the basic system of neighborhoods of zero for $\mathcal{T}_0$ or $\mathcal{T}_1$ given by (1.19). It is clearly sufficient to show that for all $n\geqq 1$,

(8.12) $$U_n \subseteq V_n.$$

To see that (8.12) holds, let x be any element of U_n. Then as $U_n = a_n U$, $x = a_n x_0$ for some x_0 in U. Then x_0 is in S_k for some k. Clearly we may take k so that $k > n$. Now by (8.11), $(a_{k-1}/a_k)x_0 \in B_k$, so by (1.5'),

$$x_0 a_{k-1} = x_0(a_{k-1}/a_k)a_k \in B_k \cdot V_k \subseteq V_{k-1}.$$

As $(a_{k-2}/a_{k-1}) \in B_{k-1}$ by (8.11),

$$x_0 a_{k-2} = (a_{k-2}/a_{k-1})x_0 a_{k-1} \in B_{k-1} \cdot V_{k-1} \subseteq V_{k-2}.$$

Repeating this procedure, we see by induction that $x_0 a_{k-j} \in V_{k-j}$, for $1 \leqq j \leqq k$, and so in particular,

$$x = x_0 a_n = x_0 a_{k-(k-n)} \in V_{k-(k-n)} = V_n.$$

This shows that (8.12) holds for all $n \geqq 1$, and so it follows that $\mathscr{T}_0 \subseteq \mathscr{T}$ and $\mathscr{T}_1 \subseteq \mathscr{T}$. Thus, $\mathscr{T}$ is the inductive ring topology (inductive locally bounded ring topology) determined by the sequences (a_k) and (B_k).

Corollary 8.5. *An inductive locally bounded ring topology on a field is an inductive ring topology.*

We note that one could give a simpler proof of Theorem 8.4 by defining the sequence (a_k) as was done and observing that $\mathscr{T} = \mathscr{T}((a_k), (C_k))$, where $C_k = U$ for each k. However, when K is a countable field, it would be of some interest to know if a ring topology $\mathscr{T}$ is an inductive ring topology $\mathscr{T}((a_k), (B_k))$, where the sets B_k are finite. We see from (8.11) that if U is countable, (i.e., K is countable), then we may take the sets B_k to all be finite. This leads us to make the following definition.

Definition. Any of the inductive topologies $\mathscr{T}((a_k), (B_k))$ or $\mathscr{T}_b((a_k), (B_k))$ will be called *finitely generated* if each of the sets B_k is finite.

We are able to get, then, from Theorem 8.4, the following characterization of all first countable, locally bounded ring topologies on a countable field K.

Corollary 8.6. *The class of all first countable, locally bounded ring topologies on a countable field K is precisely the class of all finitely generated, inductive locally bounded ring topologies.*

It is easily seen that if the sets B_n are all finite, then the sets V_n^m given by (1.18) are also finite. Thus, Corollary 8.6 gives an effective method for approximating a basic system of neighborhoods of zero for any first countable, locally bounded ring topology on any countable field.

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